Worksheet 3

The bracket associates to each unoriented link diagram D a Laurent polynomial $\langle D\rangle \in \mathbf{Z}[q, q^{-1}]$. It is characterized by $\langle\text{crossing}\rangle = \langle\text{0-smoothing}\rangle - q\langle\text{1-smoothing}\rangle$ and $\langle C\rangle = (q + q^{-1})^k$ where C is a crossingless diagram with k circles. The *unnormalized* Jones polynomial of an oriented link K is $J(K) = (-1)^{n_-} q^{n_+ - 2n_-}\langle D\rangle$. The *normalized* Jones polynomial is $V(K) = J(K)/(q + q^{-1})$. Note that $V(U) = 1$ while $J(U) = q + q^{-1}$ where U is the unknot. The polynomial V is typically what is referred to as the Jones polynomial, often with the convention that $q = -t^{1/2}$. The polynomial J will be the Euler characteristic of (unreduced) Khovanov homology.

Problem 1. Let D be an unoriented link diagram with n crossings. Show that

$$\langle D\rangle = \sum_{k=0}^{n} (-q)^k \sum_{C} (q + q^{-1})^{\#C}$$

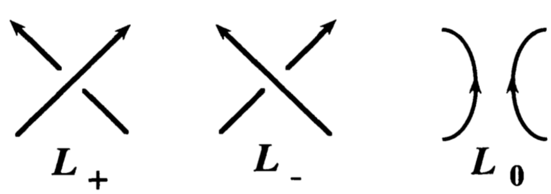
where the second sum is over all crossingless diagrams C obtained by smoothing D using k 1-smoothings and $n - k$ 0-smoothings. The nonnegative integer $\#C$ is the number of circles appearing in the crossingless diagram C.

Problem 2. Show that if $m(L)$ denotes the mirror of the oriented link L, then $V(m(L))$ is obtained from $V(L)$ by replacing q by q^{-1}. Show that the right- and left-handed trefoils are distinct, and verify that the Jones polynomial of the figure-eight knot is invariant under the operation $q \mapsto q^{-1}$. What happens to the Jones polynomial if the orientation of each component of the link is reversed?

Problem 3. Compute $J(K \coprod L)$ in terms of $J(K)$ and $J(L)$ where K and L are oriented links and $K \coprod L$ is their disjoint union, separated by a plane. Compute $V(K\#L)$ in terms of $V(K)$ and $V(L)$ where K and L are oriented *knots*. What about oriented links? Can you find a pair of distinct links with the same Jones polynomial V?

Problem 4. The Skein relation.

(a) Let L_+, L_-, L_0 denote oriented links which are related by the following local modifications



Establish the "Skein relation":

$$q^{-2}V(L_+) - q^2 V(L_-) = (q^{-1} - q)V(L_0).$$

(b) Let D be an oriented diagram of a link. Show that there is a sequence of crossing changes of D which produces a diagram for the unlink.

(c) Prove that any $\mathbf{Z}[q, q^{-1}]$-valued oriented link invariant V^* which satisfies $V^*(U) = 1$ and the Skein relation must equal the Jones polynomial V.

(d) Use the Skein relation to compute the Jones polynomial of the "cinquefoil," also known as 5_1 or the $(2, 5)$ torus knot. Is it isotopic to its mirror?

Problem 5. Mutation-invariance of the Jones polynomial.

Given a diagram D for an oriented link, find a circle in the plane which intersects the diagram in exactly four points, which we may assume are equally spaced out on the circle and one lies at the angle 45° with respect to given x and y axes. Rotate the portion of the knot within the circle 180° about one of the three axes (the x or y axes or the z axis perpendicular to the page) so that the endpoints match up again to form a link diagram D'. For example, the Kinoshita-Terasaka knot and the Conway knot:

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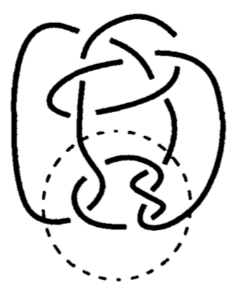

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(a) Show that exactly one of the three possible resulting diagrams naturally inherits a coherent orientation. Show that the other two diagrams are oriented after changing the orientation on all of the strands within the circle.

With the orientation described in part (a), the resulting link is called a *mutant* of the original link.

(b) Show that the crossings of D lying within the circle are in a natural one-to-one correspondence with the crossings of D' lying within the circle in such a way that positive crossings in D correspond to positive crossings in D'. In particular, the two diagrams D and D' have the same n_+ and the same n_-.

(c) Choose a smoothing (0- or 1-) for each crossing of D, and apply the smoothings to obtain a crossingless diagram. Show that one obtains a crossingless diagram with the same number of components by applying the same smoothings for the crossings of D' under the above one-to-one correspondence of crossings within the circles (and the obvious one-to-one correspondence of crossings outside the circles). *Hint:* analyze the possible crossingless diagrams within the circle.

(d) Show that the Jones polynomial is mutation-invariant, i.e. if L and L' are mutants, then $J(L) = J(L')$.

Remark. Sometimes this fact is thought of as a failure of the Jones polynomial to distinguish between mutants. However, it can also be thought of as the ability to show that, for example, the trefoil is not a mutant of the figure eight, or more generally that knots with distinct Jones polynomials are not mutants.

Remark. The Kinoshita-Terasaka knot and the Conway knot are the most famous pair of mutants. Their Jones polynomials are identical, and their Alexander polynomials are both equal to 1. Their Khovanov homologies are also isomorphic. However, it turns out that the Kinoshita-Terasaka knot is of genus 2 while the Conway knot is of genus 3.