Worksheet 9

July 22, 2019

1. (a) Explain why every knot can be considered as a ribbon graph (consisting of only one copy of $S^1 \times [0,1]$ embedded in $\mathbb{R}^3$ between the planes $z = 1$ and $z = 0$.)

   (b) Explain in what sense a graph is a ribbon graph (don't try to make this part too precise/ "functorial").

2. **(Justification of motivation from last time.)** Recall that an $n+1$-dimensional cobordism between two n-manifolds is an $n+1$-dimensional manifold C with $\partial C = M \sqcup N$. Explain how each ribbon graph gives rise to a 1-dimensional cobordism between 0-manifolds (i.e. a collections of points).

3. **(Bialgebras and Hopf algebras.)** A bialgebra over a field k is a vector space over k with multiplication and comulitplication, compatible in an appropriate sense. More precisely, a **bialgebra** H is an algebra object in the opposite category of Alg_k, for k a field. The bialgebra is said to be a **Hopf algebra** if it equipped with a k-linear anti-homomorphism S, called the **antipode**. That is, a k-linear map $S \colon H \to H$ so that

$$S(x, y) = S(y)S(x).$$

   (a) Explicitly specify the other data involved in the definition of a bialgebra: multiplication, comultiplication, unit, and counit maps. Then write out commutative diagrams that relating these maps.

   (b) If you are feeling ambitious, try to come up with a reasonable diagram to impose on the antipode map S!

   Remark 1. *The antipode map is often suppressed in discussions of specific Hopf algebras. We will discuss the antipode map in class as needed. Intuitively, it should be thought of as a substitute for the map* *sending x to x^{-1} (which does not in general exist, since most elements in a general bialgebra do not have multiplicative inverses! This idea should help you come up with the diagram for part (b).*

(c) Think of some examples of bialgebras and/or Hopf algebras.

4. Check that the "braiding" on $HDCR(A)$ (as defined in class) does indeed endow the category with a braided monoidal structure.

5. Check that the definition of duals, evaluation, and coevaluation (as defined in class) does indeed make $HCDR(A)$ into a compact, braided monoidal category.

6. In class, we worked primarily with the case of **homogeneous** ribbon graphs (those which meet the planes $z = 1$ and $z = 0$ with the "white" side of the graph facing "up". All constructions discussed also make sense for non-homogeneous graphs, with a few modifications: essentially, we need to keep track of which "side" is up. We define a category $CDR(A)$ with:

   - Objects finite sequences $\big((V_i, \epsilon_i, \nu_i)\big)_{1\le i\le k}$, where k varies. As before, $V_i \in \Lambda(A)$, and $\epsilon_i \in \{\pm 1\}$ correspond to orientation of cores of ribbons. ν_i is also either 1 or -1, and should be thought of as encoding a prescribed orientation at a given base (with $+1$ being white and -1 being black).
   - A morphism

$$\big((V_i, \epsilon_i, \nu_i)\big)_{1\le i\le k} \to \big((V'_i, \epsilon'_i, \nu'_j)\big)_{1\le j\le m}$$

     is a (not necessarily homogeneous) CDR-graph Γ with associated sequence of bottom (resp. top) colors $(V_1, \dots, V_k)$ (resp. with m instead of k and primes), bottom (resp. top) "core directions" equal to $(\epsilon_1, \dots, \epsilon_k)$ (resp. with m instead of k, and primes), and bottom (resp. top) orientations being $(\nu_1, \dots, \nu_k)$ (resp. with m instead of k and primes).

   Remark 2. *Note that, pictorially, morphisms in this category go up: from the $z = 0$ plane to the $z = 1$ plane. (Just as for $HCDR(A)$.)*

   Show that, with these definitions, $CDR(A)$ is also a strict monoidal category, and has a compact braided structure defined analogously to (and extending) that on $HCDR(A)$.